An efficient method to represent and process imprecise knowledge

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Abstract

We are primarily concerned with the problem of representing imprecise statements and knowledge as well as drawing conclusions based on this type of knowledge. Our particular interest is to establish an efficient method, capable to represent and apply (i.e. reason with) imprecise knowledge within real problems. In the present paper we first introduce an axiomatic framework and discuss it with illustrative examples. One suggestion for an application-oriented specialization is given by scalar fuzzy control (SFC), which is presented in the second part of this paper. After the introduction of the SFC theory, it is proofed that it is a member of the axiomatic framework. Its usage is finally illustrated by applying it to the well known inverted pendulum problem.

Keywords: Knowledge representation, imprecise knowledge, Scalar fuzzy control, SFC

1. Introduction

Logic, including multiple-valued logic, has been the subject of extensive research for decades. Starting with the early works of Boole [1], [2], “classical" logics like propositional logic or predicate logic are two-valued logics and thus primarily valuable to represent precise knowledge, while the aim of many-valued logic (MVL), including infinite-valued logic, is to provide formal representation languages which can be used to represent precise as well as imprecise knowledge. [8] tried to produce a syntax for the logic of vague concepts, inspired by Lukasiewicz logic and based on MVL. In [12] Hurst gives a good overview of the historical development of MVL, including circuit realizations, while [11] gives an overview of many of the theoretical works on MVL, including a classification of the different approaches. Possibilistic logic [7] focuses on expressing uncertain knowledge in terms of possibility-qualified statements. It treats syntactic objects, expressing inequalities resulting from statements.

Fuzzy Set Theory [24] and especially Fuzzy Control is often used to represent non-dichotomous knowledge. Besides the possibility to use fuzzy sets in inference processes (denominated as *approximate reasoning*) [25], Fuzzy Set Theory also can be used to draw conclusions from (imprecise) premises which are not identical to the antecedents of the knowledge base (denominated as *plausible reasoning*) [28]. Huge fuzzy knowledge bases can require too much computing power, however, especially if the linguistic variables include many terms. [6] and [10] give a survey on these problems in their first section, including a summary on suggested approaches to solve these problems. Another problem is that the implementation depends on the definition of *suitable* linguistic variables and fuzzy sets as well as on the choice of the appropriate inference method, operators etc. It is not always apparent how the number and shape of fuzzy sets will effect the result of the inference process, especially because it might depend on the selected inference method. On the other hand the inference method to be selected depends on the modeling of the imprecise knowledge, since not all fuzzy inference methods are suitable for all kinds of problems (see, for example [5], [21], [26], [27]).

Our interest is to develop an efficient method which is suitable to represent imprecise knowledge as well as the reasoning processes as such and which is simple enough to have a high degree of practical usability to solve real world problems. We thus start with the question of how human beings express (i.e. verbalize) imprecise knowledge.

A typical method to formulate knowledge are IF-THEN statements. The IF-condition can either be true or false; other

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states are not possible in two valued logic. The same holds for the THEN-conclusion, which – depending on the state of the IF-condition – is either true or not. It is thus difficult to represent imprecise knowledge with simple IF-THEN statements. Often it is reasonable to represent imprecise knowledge in terms of THE-MORE-THE statements rather than in terms of IF-THEN statements, because the first in certain situations is closer to the linguistic verbalization of knowledge and thus simplifies knowledge implementation, validation and application.

As an example, if we ask a person (who is not familiar with mathematical logic) how he would slow down his car if a very slow car is in front of him, he might answer "If the distance to the car in front is too short, I will start braking". This statement is incomplete and does not reflect the complete knowledge of that person. Asking to specify "start braking" and maybe additionally "How do you proceed with *start braking*, and how much would you brake?", the person would probably answer something like "Ok, the shorter the distance, the more I will slow down my car". We consider this as an imprecise statement. It clearly states the *relation* between distances and decelerating force without specifying it in detail, e.g. as a functional description. The reason is simply that the human "expert" is typically not able to provide a mathematical or functional description of his knowledge. Nevertheless he is able to formulate his knowledge and apply it to solve the problem perfectly.

Even *plausible reasoning* is not ideal to express imprecision in THE-MORE-THE statements. Hence we propose in this paper a new framework to handle this kind of reasoning.

In Section 2 we will define the framework by its notions and axioms. Section 3 will discuss the representation of imprecise knowledge, while Section 4 focuses on the characteristics of the new method presented in this paper. In Section 5 we discuss differences to mathematical logics such as two-valued logic. While the axiomatic framework gives a formal definition only, one suggestion for an application oriented specialization is given by scalar fuzzy control (SFC) [16]. Section 6 gives an introduction to this approach and proves that it is a member of the framework defined in Section 2. Section 7 illustrates how to use SFC by applying it to the well known inverted pendulum problem.

2. An axiomatic framework for representing imprecise Knowledge

2.1. Definitions

We introduce the framework by defining the following notions:

- truth values
- constants
- variables
- operators
- terms
- predicates
- conjunctors
- formula

Definition 1. A *truth value* is defined as a real value lying in the unit interval [0,1].
The extremes 0 and 1 shall be interpreted as "completely false" and "completely true" respectively, while values in between 0 and 1 indicate partial truth.

The unit interval was chosen according to Boolean logic and algebra, where the two symbols "0" and "1" are used as the two truth values.

Very often the "degree of truth" is expressed by choosing an appropriate truth value as a real number lying on the unit interval. Here statements are represented in terms of imprecise relations. Hence, imprecision is determined by the evaluation of the imprecise relations directly.

Definition 2. All real numbers $z \in \Re$ are *constants.*

Definition 3. v is a *variable* if and only if its value r is a real numbers, i.e. $r \in \Re$.

Definition 4. *Operators* are all algebraic one-place and multiple-place operators.

All operators known from algebra like addition, subtraction etc. as well as trigonometric functions etc. are valid operators.

Definition 5. *Terms* are defined by:

1. A constant is a term.
2. A variable is a term.
3. If t_1 ... t_n with $n \geq 1$ are terms and f is a n-place operator, then $f(t_1, ..., t_n)$ is a term too.
4. No other terms exist.

In order to make it possible to represent THE-MORE-THE statements we now have to introduce a new class of predicates, which differ from the classical predicates such as "less", "greater", etc. Since they are derived from these, however, we call them *extended predicates.*

Definition 6. The following *extended predicates* are defined for all terms x, y:

extended equal:	$E_e(x, y)$	$: \Re \times \Re \to [0,1]$
extended not equal:	$NE_e(x, y)$	$: \Re \times \Re \to [0,1]$
extended less:	$L_e(x, y)$	$: \Re \times \Re \to [0,1]$
extended greater:	$G_e(x, y)$	$: \Re \times \Re \to [0,1]$

The following notions will be used for the extend predicates:

extended equal:	$x \overset{e}{=} y$
extended not equal:	$x \overset{e}{\neq} y$
extended less:	$x \overset{e}{<} y$
extended greater:	$x \overset{e}{>} y$

According to the extended predicates we introduce extended conjunctors to represent imprecise combinations:

Definition 7. The following *extended conjunctors* are defined for truth values w, respectively w_i, i=1..n:

extended NOT: $\mathrm{NOT_e}(w)$: $[0,1] \rightarrow [0,1]$
averaging AND: $\mathrm{AND_{av}}(w_1, \ldots, w_n)$: $[0,1]^n \rightarrow [0,1]$
averaging OR: $\mathrm{OR_{av}}(w_1, \ldots, w_n)$: $[0,1]^n \rightarrow [0,1]$

Alternative notions used below are:

extended NOT: $\overset{e}{\neg}\,(w)$
averaging AND: $w_1 \overset{av}{\wedge} w_2 \ldots \overset{av}{\wedge} w_n$
averaging OR: $w_1 \overset{av}{\vee} w_2 \ldots \overset{av}{\vee} w_n$

Finally we define formulas:

Definition 8. *Formulas* are defined by:

1. Any truth value $w \in [0,1]$ is a formula.
2. If $t_1 \ldots t_n$ with $n\geq 1$ are terms and p is a n-place predicate, then $p(t_1, \ldots, t_n)$ is a formula too.
3. If $f_1 \ldots f_n$ with $n\geq 1$ are formulas and j is a n-place conjunctor, then $j(f_1, \ldots, f_n)$ is a formula too.
4. No other formula exist.

2.2. Axioms

We define the framework by a set of axioms:

Definition 9. A mathematical method according to Definitions 1 through 8 is called *Calculus of Imprecise Knowledge* if and only if the following axioms are satisfied.

Axioms for the extended predicates:

For all $x, y \in \Re$:

P1) The predicates $\mathrm{E_e}$ and $\mathrm{NE_e}$ are commutative, i.e.
$\mathrm{E_e}(x, y) \equiv \mathrm{E_e}(y, x)$
$\mathrm{NE_e}(x, y) \equiv \mathrm{NE_e}(y, x)$
The predicates $\mathrm{L_e}$ and $\mathrm{G_e}$ are reciprocal commutative, i.e.
$\mathrm{L_e}(x, y) \equiv \mathrm{G_e}(y, x)$

P2) The predicates $\mathrm{E_e}$, $\mathrm{NE_e}$, $\mathrm{L_e}$ and $\mathrm{G_e}$ are symmetric in the following sense:
$\mathrm{E_e}(y-a, y) \equiv \mathrm{E_e}(y+a, y)$
$\mathrm{NE_e}(y-a, y) \equiv \mathrm{NE_e}(y+a, y)$
$\mathrm{L_e}(y-a, y) \equiv 1 - \mathrm{L_e}(y+a, y)$
$\mathrm{G_e}(y-a, y) \equiv 1 - \mathrm{G_e}(y+a, y)$

P3) The predicates $\mathrm{E_e}$, $\mathrm{NE_e}$, $\mathrm{L_e}$ and $\mathrm{G_e}$ are symmetric in the following sense:
$\mathrm{E_e}(x, y) \equiv 1 - \mathrm{NE_e}(y, x)$
$\mathrm{L_e}(x, y) \equiv 1 - \mathrm{L_e}(y, x)$
$\mathrm{G_e}(x, y) \equiv 1 - \mathrm{G_e}(y, x)$

P4) The predicates $\mathrm{E_e}$, $\mathrm{NE_e}$, $\mathrm{L_e}$ and $\mathrm{G_e}$ are reflexive in the following sense:
$\mathrm{E_e}(x, x) \equiv 1$ (x is identical to x)
$\mathrm{NE_e}(x, x) \equiv 0$ (x is identical to x)
$\mathrm{L_e}(x, x) \equiv 0.5$ (x is neither greater nor less than x)
$\mathrm{G_e}(x, x) \equiv 0.5$ (x is neither greater nor less than x)

P5) For fixed y and $x \in \Re$:
$\mathrm{E_e}(x, y)$ is strictly monotonically increasing for $x \leq y$
$\mathrm{E_e}(x, y)$ is strictly monotonically decreasing for $x \geq y$
$\mathrm{NE_e}(x, y)$ is strictly monotonically decreasing for $x \leq y$
$\mathrm{NE_e}(x, y)$ is strictly monotonically increasing for $x \geq y$
$\mathrm{L_e}(x, y)$ is strictly monotonically decreasing
$\mathrm{G_e}(x, y)$ is strictly monotonically increasing

For fixed x and $y \in \Re$:
$\mathrm{E_e}(x, y)$ is strictly monotonically decreasing for $x \leq y$
$\mathrm{E_e}(x, y)$ is strictly monotonically increasing for $x \geq y$
$\mathrm{NE_e}(x, y)$ is strictly monotonically increasing for $x \leq y$
$\mathrm{NE_e}(x, y)$ is strictly monotonically decreasing for $x \geq y$
$\mathrm{L_e}(x, y)$ is strictly monotonically increasing
$\mathrm{G_e}(x, y)$ is strictly monotonically decreasing

P6) All predicates are continuously differentiable functions in $\Re^2$.

Axioms for the extended conjunctors:

For all $x, y \in \Re$:

J1) The extended NOT is defined by:
$\mathrm{NOT_e}(w) = 1 - \mathrm{w}$

J2) The conjunctors $\mathrm{AND_{av}}$ and $\mathrm{OR_{av}}$ are commutative, i.e.
$\mathrm{AND_{av}}(w_1,w_2,\ldots w_n) = \mathrm{AND_{av}}(w_2,w_1,\ldots w_n) = \ldots$
$\mathrm{OR_{av}}(w_1,w_2,\ldots w_n) = \mathrm{OR_{av}}(w_2,w_1,\ldots w_n) = \ldots$

J3) The conjunctors $\mathrm{AND_{av}}$ and $\mathrm{OR_{av}}$ have an "averaging" characteristic, i.e.
$\mathrm{AND_{av}}(w_1,w_2,\ldots w_n) = \min(w_2,w_1,\ldots w_n)+\Delta_{\mathrm{ANDav}}$
$\mathrm{OR_{av}}(w_1,w_2,\ldots w_n) = \max(w_2,w_1,\ldots w_n)-\Delta_{\mathrm{ORav}}$
with:
1. $\Delta_{\mathrm{ANDav}} = \Delta_{\mathrm{ANDav}}(w_1, w_2, \ldots w_n)$
   $\Delta_{\mathrm{ORav}} = \Delta_{\mathrm{ORav}}(w_1, w_2, \ldots w_n)$
2. For $w_1=w_2=\ldots=w_{\mathrm{i}}$:
   $\Delta_{\mathrm{ANDav}} = 0$ and $\Delta_{\mathrm{ORav}} = 0$
   all other cases:
   $0 < \Delta_{\mathrm{ANDav}} < \max(w_1,w_2,\ldots w_n)-\min(w_1,w_2,\ldots w_n)$
   $0 < \Delta_{\mathrm{ORav}} < \max(w_1,w_2,\ldots w_n)-\min(w_1,w_2,\ldots w_n)$
3. $\Delta_{\mathrm{ANDav}}(w_1,\ldots,w_{\mathrm{i}},\ldots w_n)<\Delta_{\mathrm{ANDav}}(w_1,\ldots,w_{\mathrm{i}}+\delta,\ldots w_n)$
   for $w_{\mathrm{i}} \neq \min(w_1, w_2, \ldots w_n)$, $\mathrm{i} \in \{1\ldots\mathrm{n}\}$
   and $w_{\mathrm{i}} < w_{\mathrm{i}} + \delta \leq 1$
   $\Delta_{\mathrm{ORav}}(w_1,\ldots,w_{\mathrm{i}},\ldots w_n)<\Delta_{\mathrm{ORav}}(w_1,\ldots,w_{\mathrm{i}}-\delta,\ldots w_n)$
   for $w_{\mathrm{i}} \neq \max(w_1, w_2, \ldots w_n)$, $\mathrm{i} \in \{1\ldots\mathrm{n}\}$
   and $0 \leq w_{\mathrm{i}} - \delta < w_{\mathrm{i}}$

J4) The conjunctors $\mathrm{AND_{av}}$ and $\mathrm{OR_{av}}$ are continuous functions in $\Re^{\mathrm{n}}$.

Axiom P1 claims that the predicates E_e and NE_e are commutative, which is quite obvious: if we say that x is similar to y up to a certain degree, y should be similar to x to the same extent. A similar intention underlies axioms P2 and P3 which claims a symmetric behavior of the predicates. Notice that both use the extended NOT of axiom J1. Axiom P4 claims the reflexivity which is also obvious. As an example, x is in fact equal to x, so the result of the extended predicate E_e should be identical to the result of the corresponding Boolean predicate. Axioms P5 and P6 claim that the extended predicates are monotonic and continuously differentiable functions. This is important in order to receive smooth solution spaces when applying the extended predicates to solve problems. As an example, using extended predicates with non monotonic and/or non-steady behavior as well as a non-steady derivation in control applications can lead to oscillating and/or unstable system behavior.

Axiom J1 defines the extended NOT. As the extended predicates the extended conjunctors should be commutative too, as claimed by axiom J2. In this paper we discuss "averaging" conjunctors. So AND_{av} is always greater than the minimum, i.e. $AND_{av}(w_1, w_2)$ is *not* a t-norm [13]. Similar OR_{av} is always less than the maximum, i.e. $OR_{av}(w_1, w_2)$ is *not* an s-norm [13]. Axiom J3-1 claims that the degree of "averaging" depends on the truth values $w_1 \dots w_n$ only. From J3-2 it follows, that $AND_{av}(w_1, w_2, \dots w_n)$ will never be larger than the greatest value of $w_1 \dots w_n$ and that $OR_{av}(w_1, w_2, \dots w_n)$ will never be less than the lowest value of $w_1 \dots w_n$. From this we call the extended conjunctors *averaging* conjunctors. One might also think of a *non-averaging* AND and OR in the style of a t-norm and an s-norm, but this shall not be further discussed in this paper. J3-3 claims for AND_{av}, that if all truth values $w_1 \dots w_n$ stay the same except one, which is not the smallest one and which is increasing, than the compensation will increase too. This thus results in an increased truth value for the AND_{av} conjunction. Similar is true for OR_{av}. Finally axiom J4 again claims a continuous function in order to receive smooth solution spaces.

3. Representation of imprecise knowledge

Before considering the characteristics of the *Calculus of Imprecise Knowledge* defined above, we first want to discuss its use. Doing so we want to stress that this framework is application oriented. Furthermore, it might help the reader to understand the intention and purpose of its usage.

As already discussed, knowledge is quite often represented by IF-THEN statements. A typical inference method to apply knowledge represented in terms of IF-THEN statements is Modus Ponens. Here, a statement is implemented as an implication, using the two-place *implication conjunctor*. In the *Calculus of Imprecise Knowledge* discussed here no implication conjunctor is used. The inference method bases on the extended predicates instead. We thus avoid all problems implication conjunctors may have within many-valued logics.

We reconsider our example in the introduction and demonstrate the knowledge representation in terms of THE-MORE-THE statements:

Example 1:

THE MORE	my distance to the car in front decreases,
THE	more I will slow down my car.

Denoting the *condition* by A and the *conclusion* by B, we obtain:

THE MORE A, THE more B .

To simplify writing, we use the following notation

$$A \overset{E}{\rightarrow} B \tag{1}$$

where $\overset{E}{\rightarrow}$ symbolizes the *extended path of reasoning*, i.e. it is *not* an implication conjunctor.

An imprecise statement could also express a "negative" relation, such as in

Example 2:

THE MORE	I have to save money,
THE	*less* I will spend my money on buying cars

which is symbolically written as

$$A \overset{E}{\rightarrow} \left(\overset{e}{\neg} B \right) \tag{2}$$

with again A being the condition and $\overset{e}{\neg} B$ being a "negative" conclusion.

The condition A may be composed of several *antecedents*, which are "combined" by the extended conjunctors.

Given the premise $A' \neq A$ we now ask for the conclusion B', which might be close to B but not necessarily identically to B, depending on the underlying knowledge. Following the idea of the Generalized Modus Ponens [28], we introduce the *Extended Modus Ponens*:

Statement:	$A \overset{E}{\rightarrow} B$	Statement:	$A \overset{E}{\rightarrow} \overset{e}{\neg} B$
Premise:	A'	Premise:	A'
Conclusion:	B'	Conclusion:	B'

which can be symbolically written as

$$\left[A', A \overset{E}{\rightarrow} B \right] \overset{E}{\mapsto} B' \quad , \quad \left[A', A \overset{E}{\rightarrow} \overset{e}{\neg} B \right] \overset{E}{\mapsto} B' \tag{3}$$

respectively.

The advantage of the *Calculus of Imprecise Knowledge* is that the imprecise conditions and conclusions of knowledge can directly be formulated by using the extended predicates and conjunctors. To illustrate this, we will extend our Example 1 to

Example 3:

THE MORE	my distance to the car in front decreases
AND	the more my speed is high,
THE	more I will slow down my car.

which we write as

$$\left(D \overset{av}{\wedge} S \right) \overset{E}{\rightarrow} B \quad . \tag{4}$$

The next step is to specify the two antecedents *D* and *S*. Distances of 20 m or less might be understood as a short distance and speeds above 80 km/h might be high, so we would obtain

$$\left(\left(d \overset{e}{<} 20\,\mathrm{m} \right) \overset{av}{\wedge} \left(s \overset{e}{>} 80\,\mathrm{km/h} \right) \right) \overset{E}{\rightarrow} B \tag{5}$$

where *d* is the distance in m and *s* the speed in km/h. One might be confused that we state *exact* values in the antecedents (20 m, respectively, 80 km/h) while we are talking about imprecise knowledge. But in fact, when we ask people from what value they think that the distance to the front car is (too) short, most of them would reply something like "well, from *about* 20 m". Although a concrete value is given (20 m), the complete statement is imprecise, emphasized by the phrase "*about*". Psycho-linguistic examinations have shown that even for imprecise statements it is very often easy to state a concrete value, denominated as anchor-point [28]. Using *Calculus of Imprecise Knowledge* the concrete value (20 m) is used and the imprecision of the statement is implemented by the usage of the extended predicates.

The result of the imprecise comparison is a truth value, which is characterized by the definition and axioms of the extended predicate. Anticipating some results of the detailed discussion of the *Calculus of Imprecise Knowledge* in the following section, we observe that the result of any (simple or combined) condition is always a truth value $w \in [0,1]$.

To complete Example 3, we have to ask for the details of the conclusion, i.e. how much force should be used to slow down the car. We should orient our selfs on an emergency braking where the maximum decelerating force is needed, to cover all possible situations. Given that the maximum decelerating force for an emergency braking is for example 75 N, we would obtain

$$B = 75\,\mathrm{N} \cdot \left(\left(d \overset{e}{<} 20\,\mathrm{m} \right) \overset{av}{\wedge} \left(s \overset{e}{>} 80\,\mathrm{km/h} \right) \right) \tag{6}$$

which is the representation of our imprecise knowledge. Using *Extended Modus Ponens*, we now have the premises *d*' (actual distance) and *s*' (actual speed). The corresponding decelerating force *B*' is calculated by simply substituting *d* by *d*' and *s* by *s*', i.e.

$$B' = 75\,\mathrm{N} \cdot \left(\left(d' \overset{e}{<} 20\,\mathrm{m} \right) \overset{av}{\wedge} \left(s' \overset{e}{>} 80\,\mathrm{km/h} \right) \right) \quad . \tag{7}$$

From this, the extended Modus Ponens is realized by directly applying the actual values to the implemented imprecise statements.

A yet open question is a functional definition of the extended predicates and conjunctors. While the *Calculus of Imprecise Knowledge* gives an axiomatic definition only, one suggestion for a concrete implementation is given by scalar fuzzy control (see Section 6).

4. Characteristics of the *Calculus of Imprecise Knowledge*

The *Calculus of Imprecise Knowledge* has some important characteristics. We start by discussing the extended predicates, followed by the extended conjunctors and combinations of both. All theorems are discussed to facilitate the understanding and show their relevance.

Theorem 1. The extended predicates satisfy:

$$E_e(x,y) \begin{cases} < 1.0 & \text{for x < y} \\ = 1.0 & \text{for x = y} \\ < 1.0 & \text{for x > y} \end{cases}$$

$$NE_e(x,y) \begin{cases} > 1.0 & \text{for x < y} \\ = 0.0 & \text{for x = y} \\ > 1.0 & \text{for x > y} \end{cases}$$

$$L_e(x,y) \begin{cases} > 0.5 & \text{for x < y} \\ = 0.5 & \text{for x = y} \\ < 0.5 & \text{for x > y} \end{cases}$$

$$G_e(x,y) \begin{cases} < 0.5 & \text{for x < y} \\ = 0.5 & \text{for x = y} \\ > 0.5 & \text{for x > y} \end{cases}$$

Proof:

1. For $x = y$ the assertion is given by axiom P4.
2. For $x \neq y$:
   a) $\mathrm{E_e}(x,y) \underset{(P5)}{<} \mathrm{E_e}(y,y) \underset{(P4)}{=} 1.0$

   for $x < y$ as well as $x > y$

   b) $\mathrm{NE_e}(x,y) \underset{(P5)}{>} \mathrm{NE_e}(y,y) \underset{(P4)}{=} 0.0$

   for $x < y$ as well as $x > y$

   c) $\mathrm{L_e}(x,y) \underset{(P5)}{>} \mathrm{L_e}(y,y) \underset{(P4)}{=} 0.5$

   for $x < y$

   $\mathrm{L_e}(x,y) \underset{(P5)}{<} \mathrm{L_e}(y,y) \underset{(P4)}{=} 0.5$

   for $x > y$

   d) $\mathrm{G_e}(x,y) \underset{(P5)}{<} \mathrm{G_e}(y,y) \underset{(P4)}{=} 0.5$

   for $x < y$

   $\mathrm{G_e}(x,y) \underset{(P5)}{>} \mathrm{L_e}(y,y) \underset{(P4)}{=} 0.5$

   for $x > y$ □

In Example 3, where the imprecise statement "the more the distance decreases" is represented by

$$\left(d \overset{e}{<} 20\,\mathrm{m} \right) \tag{8}$$

Theorem 1 states, that given the premise *d*' = 20m the truth value of the imprecise statement is 0.5, which is neither true nor false and thus results in a "mediocre" conclusion. It is important to understand this relation between the terms and the extended predicate. In other words, when refining imprecise knowledge one should *not* think of the distance where one would very, very slightly start to break, nor of the distance where an emergency breaking is necessary, but of the distance where a *moderate* deceleration would be advisable. In fact, this is what typically is replied when asking from what value the distance to the front car is supposed to be too short and braking is advisable: Most people ask themselves, at what distance they would *moderately* brake instead of when they would slightly touch the brake pedal or when they would perform an emergency braking.

Theorem 2. For $|x - y| \to \infty$ the extended predicates behave like the Boolean predicates, i.e.:

$$\lim_{|x-y|\to\infty} E_e(x, y) = 0$$

$$\lim_{|x-y|\to\infty} NE_e(x, y) = 1$$

$$\lim_{(x-y)\to+\infty} L_e(x, y) = 0 \quad \text{and} \quad \lim_{(x-y)\to-\infty} L_e(x, y) = 1$$

$$\lim_{(x-y)\to+\infty} G_e(x, y) = 1 \quad \text{and} \quad \lim_{(x-y)\to-\infty} G_e(x, y) = 0$$

Proof:

1. According to Definition 6 it is necessary that $0 \le E_e(x, y) \le 1$
   For fixed y and $x\to\pm\infty$ from axiom P5 follows: $E_e(x, y) \to 0$
   For fixed x and $y\to\pm\infty$ from axiom P5 follows: $E_e(x, y) \to 0$
2. According to Definition 6 it is necessary that $0 \le NE_e(x, y) \le 1$
   For fixed y and $x\to\pm\infty$ from axiom P5 follows: $NE_e(x,y)\to1$
   For fixed x and $y\to\pm\infty$ from axiom P5 follows: $NE_e(x,y)\to1$
3. According to Definition 6 it is necessary that $0 \le L_e(x, y) \le 1$
   For fixed y and $x\to+\infty$ from axiom P5 follows: $L_e(x, y) \to 0$
   For fixed y and $x\to-\infty$ from axiom P5 follows: $L_e(x, y) \to 1$
   For fixed x and $y\to+\infty$ from axiom P5 follows: $L_e(x, y) \to 1$
   For fixed x and $y\to-\infty$ from axiom P5 follows: $L_e(x, y) \to 0$
4. According to Definition 6 it is necessary that $0 \le G_e(x, y) \le 1$
   For fixed y and $x\to+\infty$ from axiom P5 follows: $G_e(x,y) \to 1$
   For fixed y and $x\to-\infty$ from axiom P5 follows: $G_e(x,y) \to 0$
   For fixed x and $y\to+\infty$ from axiom P5 follows: $G_e(x,y) \to 0$
   For fixed x and $y\to-\infty$ from axiom P5 follows: $G_e(x,y) \to 1$ □

Theorem 2 states, that if premise and antecedent are "totally" different, than the imprecise comparison (represented by an extended predicate) becomes "totally" true respectively "totally" false. This is also an important characteristic, because it reflects a basic nature of imprecise knowledge. In the Example 3 it is thus possible that we obtain *no braking at all*, if the distance to the car in front becomes big enough, which is clearly reasonable and represents the human intention.

Theorem 3. The extended predicates satisfy:

$$E_e(x, y) \equiv E_e(x+a, y+a)$$
$$NE_e(x, y) \equiv NE_e(x+a, y+a)$$
$$L_e(x, y) \equiv L_e(x+a, y+a)$$
$$G_e(x, y) \equiv G_e(x+a, y+a)$$

Proof:

1. $$E_e(x, y) \underset{(x=y-a)}{=} E_e(y-a, y) \underset{(P2)}{=} E_e(y+a, y) \underset{(y=x+a)}{=} E_e(y+a, x+a) \underset{(P1)}{=} E_e(x+a, y+a)$$
2. $$NE_e(x, y) \underset{(x=y-a)}{=} NE_e(y-a, y) \underset{(P2)}{=} NE_e(y+a, y) \underset{(y=x+a)}{=} NE_e(y+a, x+a) \underset{(P1)}{=} NE_e(x+a, y+a)$$
3. $$L_e(x, y) \underset{(x=y-a)}{=} L_e(y-a, y) \underset{(P2)}{=} 1 - L_e(y+a, y) \underset{(y=x+a)}{=} 1 - L_e(y+a, x+a) \underset{(P3)}{=} L_e(x+a, y+a)$$
4. $$G_e(x, y) \underset{(x=y-a)}{=} G_e(y-a, y) \underset{(P2)}{=} 1 - G_e(y+a, y) \underset{(y=x+a)}{=} 1 - G_e(y+a, x+a) \underset{(P3)}{=} G_e(x+a, y+a)$$ □

This characteristic might lead to confusion, because someone could think that it prevents a reasonable representation of imprecise knowledge. In Example 3, if we have a premise of d'=20 m using formula (8) results in a truth value of 0.5, which is reasonable. If we now add 1080 m to both parts of the antecedent we obtain

$$\left(d_2 \overset{e}{<} 2000\,\mathrm{m} \right) \quad (9)$$

as the imprecise statement. For a premise d_2' = 2000 m, the truth value of the imprecise statement is again 0.5. This may not sound reasonable, because a moderate braking (which would result from a truth value 0.5) at a distance of 2000 m is clearly to much; no braking at all would be advisable instead. The mistake is that formula (9) does *not* represent the imprecise knowledge of our car-example any more, because we have changed the antecedent – the value of 2000m is much to high for a car. But if we transfer our knowledge to a similar, yet slightly different problem, e.g. braking a medium-sized ship, a distance of 2000 m might be suitable for a moderate braking. Theorem 3 states this characteristic: the extended predicates "behave" in the very same manner for our "ship-problem" as for our "car-problem".

Theorem 4. "Extended transitivity"; the extended predicates satisfy:

1. From $\left(x \overset{e}{<} y\right) \geq 0.5$ and $\left(y \overset{e}{<} z\right) \geq 0.5$

   follows $\left(x \overset{e}{<} z\right) \geq 0.5$
2. From $\left(x \overset{e}{>} y\right) \geq 0.5$ and $\left(y \overset{e}{>} z\right) \geq 0.5$

   follows $\left(x \overset{e}{>} z\right) \geq 0.5$

Proof:

1. $L_e(x, y) \geq 0.5 \;\Rightarrow\; x \leq y$
   $L_e(y, z) \geq 0.5 \;\Rightarrow\; y \leq z$
   Thus: $x \leq z \;\Rightarrow\; L_e(x, z) \geq 0.5$
2. $G_e(x, y) \geq 0.5 \;\Rightarrow\; x \geq y$
   $G_e(y, z) \geq 0.5 \;\Rightarrow\; y \geq z$
   Thus: $x \geq z \;\Rightarrow\; G_e(x, z) \geq 0.5$ □

Theorem 4 is of mainly theoretical interest. It demonstrates that by introducing extended predicates, also transitivity is extended.

Theorem 5. "De'Morgan law for extended predicates".

1. $\mathrm{NOT}_e\left(x \overset{e}{=} y\right) = \left(x \overset{e}{\neq} y\right)$
2. $\mathrm{NOT}_e\left(x \overset{e}{\neq} y\right) = \left(x \overset{e}{=} y\right)$
3. $\mathrm{NOT}_e\left(x \overset{e}{<} y\right) = \left(x \overset{e}{>} y\right)$
4. $\mathrm{NOT}_e\left(x \overset{e}{>} y\right) = \left(x \overset{e}{<} y\right)$

Proof:

1. $\mathrm{NOT}_e(\mathrm{E}_e(x, y)) \underset{(J5)}{=} 1 - \mathrm{E}_e(x, y) \underset{(P3)}{=} \mathrm{NE}_e(x, y)$
2. $\mathrm{NOT}_e(\mathrm{NE}_e(x, y)) \underset{(J5)}{=} 1 - \mathrm{NE}_e(x, y) \underset{(P3)}{=} \mathrm{E}_e(x, y)$
3. $\mathrm{NOT}_e(\mathrm{L}_e(x, y)) \underset{(J5)}{=} 1 - \mathrm{L}_e(x, y) \underset{(P3)}{=} \mathrm{L}_e(y, x) \underset{(P1)}{=} \mathrm{G}_e(x, y)$
4. $\mathrm{NOT}_e(\mathrm{G}_e(x, y)) \underset{(J5)}{=} 1 - \mathrm{G}_e(x, y) \underset{(P3)}{=} \mathrm{G}_e(y, x) \underset{(P1)}{=}$
   $\mathrm{L}_e(x, y)$ □

If people negate a statement, however, they do not always have the very same intention as if they would have used the opposite relation descriptor. Basically, it depends on the context if both express the very same understanding. This should be taken into consideration when acquiring knowledge and representing it by the *Calculus of Imprecise Knowledge*. Here the De' Morgan laws hold (see Theorem 9) and the double negation is the original statement, due to its basic axioms:

Theorem 6. Double negation will result in the original truth value, i.e.

$\mathrm{NOT}_e\,(\mathrm{NOT}_e\,(w)) = w$

Proof:

$\mathrm{NOT}_e\,(\mathrm{NOT}_e\,(w)) = 1 - (1 - w) = w$ □

This characteristic directly follows from axioms J1 and should be taking into consideration when applying the method.

Theorem 7. The averaging conjunctors are idempotent, i.e.:

1. $\mathrm{AND}_{av}(w, w, \ldots w) = w$
2. $\mathrm{OR}_{av}(w, w, \ldots w) = w$

Proof:

1. Due to axiom J3:
   $\mathrm{AND}_{av}(w, w, \ldots w) \underset{(J3)}{=} \min(w, w, \ldots w) + \Delta_{\mathrm{ANDav}} \underset{(J3,2.)}{=}$
   $\min(w, w, \ldots w) + 0 = w$
2. Due to axiom J3:
   $\mathrm{OR}_{av}(w, w, \ldots w) \underset{(J3)}{=} \max(w, w, \ldots w) - \Delta_{\mathrm{ORav}} \underset{(J3,2.)}{=}$
   $\max(w, w, \ldots w) - 0 = w$ □

Since the extended conjunctors are averaging conjunctors, it is reasonable that they are idempotent. It must be stressed again that averaging conjunctors are not sufficient to represent *all* styles of linguistic conjunctions, but that further conjunctors, e.g. aggregating conjunctors, are required. It will be part of our future work to integrate such additional conjunctors into the *Calculus of Imprecise Knowledge*.

Theorem 8. 0 and 1 are neutral truth values, such that:

1. $\mathrm{AND}_{av}(1,1, \ldots 1) = 1$
2. $\mathrm{AND}_{av}(0,0, \ldots 0) = 0$
3. $\mathrm{OR}_{av}\ (1,1, \ldots 1) = 1$
4. $\mathrm{OR}_{av}\ (0,0, \ldots 0) = 0$

Proof:

This characteristic directly follows from the Theorem 7. □

Theorem 8 is of mainly theoretical interest. It shows that 0 and 1 are neutral truth values as known from Boolean logic.

Theorem 9. "De'Morgan law for extended conjunctors".

Given that $\forall\, a, b \in [0,1]$

$\Delta_{\mathrm{ANDav}}(a, b) = \Delta_{\mathrm{ORav}}(1-a, 1-b)$

$\Delta_{\mathrm{ORav}}(a, b) = \Delta_{\mathrm{ANDav}}(1-a, 1-b)$

than it is necessary that:

1. $\mathrm{NOT}_e\left(a \overset{av}{\wedge} b\right) = \left(\mathrm{NOT}_e(a) \overset{av}{\vee} \mathrm{NOT}_e(b)\right)$
2. $\mathrm{NOT}_e\left(a \overset{av}{\vee} b\right) = \left(\mathrm{NOT}_e(a) \overset{av}{\wedge} \mathrm{NOT}_e(b)\right)$

Proof:

1. a) a = b

$NOT_e(AND_{av}(a, b)) =$
$1 - AND_{av}(a, b) =$
$1 - (\min(a,b) + 0) = 1 - a$
$OR_{av}(NOT_e(a), NOT_e(b)) =$
$OR_{av}((1-a), (1-b)) =$
$\max(1-a, 1-b) - 0 = 1 - a$

b) a < b
$NOT_e(AND_{av}(a, b)) =$
$1 - AND_{av}(a, b) =$
$1 - (\min(a,b) + \Delta_{ANDav}(a,b)) =$
$1 - a - \Delta_{ANDav}(a,b)$
$OR_{av}(NOT_e(a), NOT_e(b)) =$
$OR_{av}((1-a),(1-b)) =$
$\max(1-a,1-b) - \Delta_{ORav}(1-a,1-b) =$
$1 - a - \Delta_{ANDav}(a,b)$

c) a > b
$NOT_e(AND_{av}(a, b)) =$
$1 - AND_{av}(a, b) =$
$1 - (\min(a,b) + \Delta_{ANDav}(a,b)) =$
$1 - b - \Delta_{ANDav}(a,b)$
$OR_{av}(NOT_e(a), NOT_e(b)) =$
$OR_{av}((1-a),(1-b)) =$
$\max(1-a,1-b) - \Delta_{ORav}(1-a,1-b) =$
$1 - b - \Delta_{ANDav}(a,b)$

2. a) a = b
$NOT_e(OR_{av}(a, b)) =$
$1 - OR_{av}(a, b) =$
$1 - (\max(a,b) - 0) = 1 - a$
$AND_{av}(NOT_e(a), NOT_e(b)) =$
$AND_{av}((1-a), (1-b)) =$
$\min(1-a, 1-b) + 0 = 1 - a$

b) a < b
$NOT_e(OR_{av}(a, b)) =$
$1 - OR_{av}(a, b) =$
$1 - (\max(a,b) - \Delta_{ORav}(a,b)) =$
$1 - b + \Delta_{ORav}(a,b)$
$AND_{av}(NOT_e(a), NOT_e(b)) =$
$AND_{av}((1-a),(1-b)) =$
$\min(1-a,1-b) + \Delta_{ANDav}(1-a,1-b) =$
$1 - b + \Delta_{ORav}(a,b)$

c) a > b
$NOT_e(OR_{av}(a, b)) =$
$1 - OR_{av}(a, b) =$
$1 - (\max(a,b) - \Delta_{ORav}(a,b)) =$
$1 - a + \Delta_{ORav}(a,b)$
$AND_{av}(NOT_e(a), NOT_e(b)) =$
$AND_{av}((1-a),(1-b)) =$
$\min(1-a,1-b) + \Delta_{ANDav}(1-a,1-b) =$
$1 - a + \Delta_{ORav}(a,b)$ □

Theorem 9 again follows from the axioms of the imprecise knowledge. In fact, since the extended predicates follow the De' Morgan laws it is important that the extended conjunctors follow the De' Morgan laws too. Everything else could lead to confusion.

5. Differences to Mathematical Logics

In this section differences between the *Calculus of Imprecise Knowledge* presented here and mathematical logics are discussed. We will furthermore show that it is necessary and important to use a method that is different from two-valued logic if imprecise knowledge is represented.

Theorem 10. The truth value of union and intersection, respectively, of a statement and its complement can *not* be reduced to a downright truth value, i.e.:
$AND_{av}(w, NOT_e(w)) = 0$ and
$OR_{av}(w, NOT_e(w)) = 1$
are *not* satisfied for $\forall\, w \in [0,1]$!

Proof:

1. Assumption:
$AND_{av}(w, NOT_e(w)) = 0$ is true
for $\forall\, w \in [0,1]$
For $w < 0.5$ we obtain:
$AND_{av}(w, NOT_e(w)) =$
$AND_{av}(w, 1-w) =$
$\min(w, 1-w) + \Delta_{ANDav} = w + \Delta_{ANDav} > 0$
since $\Delta_{ANDav} > 0$
Contradiction $\Rightarrow$ the assumption is false.
2. Assumption:
$OR_{av}(w, NOT_e(w)) = 1$ is true
for $\forall\, w \in [0,1]$
For $w < 0.5$ we obtain:
$OR_{av}(w, NOT_e(w)) =$
$OR_{av}(w, 1-w) =$
$\max(w, 1-w) - \Delta_{ORav} = 1 - w - \Delta_{ORav} < 1$
since $\Delta_{ORav} > 0$
Contradiction $\Rightarrow$ the assumption is false. □

While the characteristic of Theorem 10 is valid for a two-valued logic, it is not meaningful if many (or infinite) truth values are used.

Imagine the statement "A tomato is red AND it is not red". Possible premises are, among other, a red tomato (truth value 1), an unripe and thus green tomato (truth value 0) or a slightly red tomato (truth value greater than 0 and less than 1). For a red tomato it is meaningful that the statement "The tomato is red AND it is not red" is "totally false". However, for a slightly red tomato it is reasonable, that the statement "The tomato is red AND it is not red" is neither "totally true" nor "totally false", but something in between.

Theorem 11.
$AND_{av}(w, 1) = w$
$OR_{av}(w, 0) = w$
are *not* satisfied for $\forall\, w \in [0,1]$!

Proof:

1. Assumption:
$AND_{av}(w, 1) = w$ is true for $\forall\, w \in [0,1]$
We have:
$AND_{av}(w, 1) = \min(w, 1) + \Delta_{ANDav} =$

$w + \Delta_{ANDav}$ with $\Delta_{ANDav}>0$ for $w \neq 1$
Contradiction ⇒ the assumption is false.

2. Assumption:
$OR_{av}(w, 0) = w$ is true for $\forall\, w \in [0,1]$
We have:
$OR_{av}(w, 0) = \max(w, 0) - \Delta_{ORav} =$
$w - \Delta_{ORav}$ with $\Delta_{ORav}>0$ for $w \neq 0$
Contradiction ⇒ the assumption is false. □

Since the extended conjunctors are supposed to be averaging conjunctors, it is directly necessary that the characteristic as stated in Theorem 11 is *not* satisfied.

In Example 3 one condition could be fulfilled to a medium degree (e.g. distance is quite low) and the other condition completely (e.g. speed is extremely high). The conclusion that the AND-combination of both should be fulfilled with the truth value of the first condition, i.e. $AND_{av}(w, 1) = w$ is definitely not what is intended, since it would result in exactly the same braking force as if the second condition would be moderately fulfilled too (speed is quite low), i.e. $AND_{av}(w, w) = w$ (see Theorem 7). It is thus meaningful, that the characteristic as stated in Theorem 11 is *not* fulfilled for the *Calculus of Imprecise Knowledge*.

Theorem 12. The extended conjunctors are *not* associative, i.e.:
$AND_{av}(a,AND_{av}(b,c)) = AND_{av}(AND_{av}(a,b),c)$
$OR_{av}(a,OR_{av}(b,c)) = OR_{av}(OR_{av}(a,b),c)$
are *not* satisfied for $\forall\, a, b, c \in [0,1]$!

Proof:

1. Assumption:
$AND_{av}(a,AND_{av}(b,c))= AND_{av}(AND_{av}(a,b),c)$
is true for $\forall\, a, b, c \in [0,1]$
For $a<b$ and $a<c$ and $b=c$ we obtain:
$AND_{av}(a,AND_{av}(b,c)) =$
$AND_{av}(a,b) =$
$\min(a,b)+\Delta_1 = a+\Delta_1$
with $\Delta_1 = \Delta_{ANDav}(a,b) > 0$
$AND_{av}(AND_{av}(a,b),c) =$
$AND_{av}(\min(a,b)+\Delta_1,c) =$
$AND_{av}(a+\Delta_1,c)$
For all $c > a+\Delta_1$ we obtain:
$AND_{av}(AND_{av}(a,b),c) =$
$\min(a+\Delta_1,c)+\Delta_2 = a+\Delta_1+\Delta_2$
with $\Delta_2 = \Delta_{ANDav}(a+\Delta_1,c) > 0$
Contradiction ⇒ the assumption is false.

2. Assumption:
$OR_{av}(a,OR_{av}(b,c)) = OR_{av}(OR_{av}(a,b),c)$
is true for $\forall\, a, b, c \in [0,1]$
For $a<b$ and $a<c$ and $b=c$ we obtain:
$OR_{av}(a,OR_{av}(b,c)) =$
$OR_{av}(a,b) =$
$\max(a,b) - \Delta_1 = b - \Delta_1$
with $\Delta_1 = \Delta_{ORav}(a,b) > 0$
$OR_{av}(OR_{av}(a,b),c) =$
$OR_{av}(\max(a,b)–\Delta_1, c) =$
$OR_{av}(b–\Delta_1, c)$
For all $c > a+\Delta_1$ we obtain:
$OR_{av}(OR_{av}(a,b),c) =$
$\max(b–\Delta_1, c) - \Delta_2 = b - \Delta_1 - \Delta_2$
with $\Delta_2 = \Delta_{ORav}(b–\Delta_1,c) > 0$
Contradiction ⇒ the assumption is false. □

Again, since the extended conjunctors are supposed to be averaging conjunctors, it is directly necessary that they are not associative. As an example, $AND_{av}(0,1)$ should be greater than 0 and $AND_{av}(d,1)$ should be greater than d, given that d is less than 1. We thus have that $AND_{av}(0,AND_{av}(1,1)) = AND_{av}(0,1)$ should be greater than 0 while $AND_{av}(AND_{av}(0,1),1) = AND_{av}(0+\Delta_1,1)$ should be greater than Δ_1 and thus greater than $AND_{av}(0,AND_{av}(1,1))$.

If we have to represent three imprecise conditions c_1, c_2 and c_3 which should be combined and the order should *not* be of any importance, than the correct representation is $AND_{av}(c_1, c_2, c_3)$ rather than $AND_{av}(c_1, AND_{av}(c_2, c_3))$.

Theorem 13. There exists no absorption law, i.e.:
$AND_{av}(a,OR_{av}(a,b)) = a$
$OR_{av}(a,AND_{av}(a,b)) = a$
are *not* satisfied for $\forall\, a, b \in [0,1]$!

Proof:

1. Assumption:
$AND_{av}(a,OR_{av}(a,b)) = a$
is true for $\forall\, a, b \in [0,1]$
For $a<b$ we obtain:
$AND_{av}(a,OR_{av}(a,b)) =$
$\min(a, \max(a,b)–\Delta_{ORav})+\Delta_{ANDav} =$
$\min(a, b–\Delta_{ORav}) + \Delta_{ANDav}$
For all $a < b–\Delta_{ORav}$ we obtain:
$AND_{av}(a,OR_{av}(a,b)) = a + \Delta_{ANDav} \neq a$
since $\Delta_{ANDav} > 0$
Contradiction ⇒ the assumption is false.

2. Assumption:
$OR_{av}(a,AND_{av}(a,b)) = a$
is true for $\forall\, a, b \in [0,1]$
For $a>b$ we obtain:
$OR_{ac}(a,AND_{ac}(a,b)) =$
$\max(a, \min(a,b)+\Delta_{ANDac})–\Delta_{ORac} =$
$\max(a, b+\Delta_{ANDac}) - \Delta_{ORac}$
For all $a > b+\Delta_{ANDac}$ we obtain:
$OR_{ac}(a,AND_{ac}(a,b)) = a - \Delta_{ORav} \neq a$
since $\Delta_{ORac} > 0$
Contradiction ⇒ the assumption is false. □

Yet, since the extended conjunctors are of averaging type and the degree of compensation depends on the parameters according to axiom J2, an absorption by means of nesting should not be possible.

This can be illustrated by the imprecise statement of

Example 4:

THE MORE I need a new car *AND* I have money available
OR the more I need a new car
THE more I should go for buying a new car

where it is obvious, that it is *not* the intention that the condition "I have money available" should be absorbed by the formulated conditions and their interrelations. Instead, there is a good reason for *not* simply stating

THE MORE I need a new car
THE more I should go for buying a new car .

The nesting of the statements reflect some "weighting". If only "I need a new car" is considerably satisfied (truth value above 0.5), than the conclusion is not that much indicated as if both, "I need a new car" *and* "I have money available" are highly satisfied.

6. Scalar fuzzy control

6.1. Method

The *Calculus of Imprecise Knowledge* is an axiomatic framework to handle imprecise knowledge. An application oriented specialization is given by scalar fuzzy control (SFC) [16], where the extended predicates and conjunctors are defined in terms of fuzzy sets. These imprecise operators can directly be applied to real-valued data, i.e. the *data* and *antecedents* are not transformed into linguistic variables and matrices. Fuzzification of (input) data and defuzzification of results is thus not required.

This section gives an introduction to SFC, including the definition and discussion of operators. We then prove in Section 6.4 that the SFC is a specialization of the *Calculus of Imprecise Knowledge* as defined in Section 2.

Using the imprecise SFC operators linguistic knowledge can directly be transformed into a mathematical representation without the need of defining multi-dimensional linguistic variables nor choosing appropriate inference and defuzzification methods. The underlying idea is, that imprecise rules such as in Example 5a may not be well represented by a set of corresponding rules as in Example 5b (Mamdami like inference), but rather by a *single* rule as in Example 5c, which is as close as possible to the original linguistic wording.

Example 5a:

The more the water is cold,
the more hot water has to be turned on

Example 5b:

IF water is very_cold
THEN hot water valve is completely_open

IF water is cold
THEN hot water valve is partly_open

IF water is medium
THEN hot water valve is half_open

IF water is hot
THEN hot water valve is partly_closed

IF water is very_hot
THEN hot water valve is completely_closed

Example 5c:

THE MORE the water is less than 40°C,
THE more hot water has to be turned on.

It must be emphasized that an *isolated* rule such as of Example 5a could be implemented by using a linguistic variable which is defined by just one fuzzy set (e.g. "cold"). In this case, just one rule would be required in Example 5b. In reality linguistic variables are often used in several up to dozens of rules and thus require to be defined by more than just one fuzzy set. Using a Mamdani like inference, several rules must be implemented then.

An implementation according to Example 5c shrinks the rulebase, because the number of rules to be implemented is reduced. The size of knowledge bases is further reduced since it is not necessary to define linguistic variables. Rather than transforming real measured values (e.g. water temperature in Example 5) to corresponding linguistic variables, the implemented rule directly uses the measured values. The imprecision of the rule is then not represented by linguistic variables, but by imprecise operators instead (in Example 5c a "Fuzzy Less" operator would be used to represent the imprecise phrase "THE MORE ... is less than ..."). This is meaningful, since in real world problems (such as in Example 5) not the measured value is imprecise (in fact it is possible to measure the water temperature with almost any desired accuracy), but the *rule* (how to act on a given temperature).

The basic approach of the *Calculus of Imprecise Knowledge* is that rules are a kind of encapsulated. If single rules have to be changed (e.g. due to extended knowledge or experience) this can be done by changing the rules' anchor-points and/or rules' operators. If one had to change fuzzy sets of linguistic variable(s), this would have side effects on other rules, using the same linguistic variables.

One might wonder whether the original rule's wording was "The more the water is *cold* ..." (Example 5a) and it is not obvious that "The more the water is *less than 40°C* ..." (Example 5c) is the correct representation, especially since a precise value (40°C) is used within an imprecise rule. In fact this is already the calibration to the underlying context of the rule to be represented. Such context depending specifications are always necessary. In fuzzy control one uses linguistic variables in which the membership functions have to be defined, i.e. one has to set up the membership functions "very_cold", "cold" etc. From this it is always necessary to define exact temperatures. Even if type-n fuzzy sets are used exact values have to be specified, because the fuzzy sets on the n^{th} level have to be defined on a base of real numbers. Typically it is much easier to specify just *one* single value (namely the anchor-point, see above) instead of several (i.e. for each fuzzy-set of the corresponding linguistic variable).

Another well known method, which also avoids defuzzification, is the Takagi-Sugeno inference. Takagi-

Sugeno requires some *analytic* knowledge about the problem to be handled, which is a drawback if such is not available. In cases as Example 5, where the knowledge can easily be put down into words but definitions of adequate formulas are not directly apparent, the *Calculus of Imprecise Knowledge* respectively SFC is a good candidate for a fast and easy realization.

6.2. Comparison operators

Comparison operators (also denominated as relation operators or predicates) compare two terms.

Definition 10. An *imprecise comparison operator* $\otimes_{cp}$ compares two real values $x,y \in \Re$:

$$\otimes_{cp}(x,y) \quad : \quad \Re \times \Re \to [0,1] \ ,$$

which may be also denoted by

$$x \otimes_{cp} y \ .$$

The result of an imprecise comparison is a *continuous* truth-value r which lies in the unit interval [0,1] and indicates the degree of truth of the imprecise comparison, where 0 indicates that the result of the comparison is "totally *not* true" and 1 indicates that the result of the comparison is "totally true". Values in between 0 and 1 indicate partial truth.

When phrasing knowledge, linguistic terms like "equal", "less" or "greater" are often used. But as already discussed a (mathematical) precision is typically *not* intended. As another example, if someone is asked to tell from which age on people are considered old, he might state an explicit age, e.g. 65 years. But actually he is *not* thinking, that all human beings up to an age of 64.999 years are not old, and starting with the 65th birthday they are old. Instead, the typical intention is that people older than *about* 65 years are old. Or more specific: the more someone is older than 65 years, the more he is considered to be old. Even for imprecise statements it is very often easy to state a specific anchor-point (65 years in the example) as psycho-linguistic examinations have shown [28].

Imprecise comparison operators provide the possibility to deal with precise values and model an imprecise process of drawing conclusions (e.g. "the more someone is older than 65 years, the more ..."). Imprecise comparison operators are fuzzy sets after all, defined by their membership function.

Since the operators shall be imprecise (i.e. non dichotomous), their shape should be smooth. To return to the example: One does not perceive that from *exactly* 65 years on a human is considered to be old. One does rather consider that from about 55 years a human starts to be slightly old and that at about 75 years the human is finally old. Getting old is a continuos process which starts slowly instead of at exact values. Reconsidering Example 5 it does in fact make sense to *not* suddenly start to turn on the hot water, but very slowly. This prevents oscillation and instability when used in closed loop controls. Furthermore the solution space of multi-dimensional rules are smooth without discontinuities.

Hence no triangular and trapezoidal membership functions are to be used for the imprecise operators, but continuously differentiable instead. We thus define the following imprecise comparison operators by:

Definition 11. For $x,\ y \in \Re$, the imprecise comparison operators

Fuzzy Equal	$FE(x,y)$	$: \ \Re \times \Re \to [0,1]$
Fuzzy Not Equal	$FNE(x,y)$	$: \ \Re \times \Re \to [0,1]$
Fuzzy Less	$FL(x,y)$	$: \ \Re \times \Re \to [0,1]$
Fuzzy Greater	$FG(x,y)$	$: \ \Re \times \Re \to [0,1]$

are defined by:

$$FE_{\Delta xy,\Delta r}(x,y) = 1 - \tanh^2\big((x-y)\cdot m\big)$$

with $\Delta xy > 0,\ 0 < \Delta r < 1.0,\ m > 0$
and $m = \operatorname{artanh}\big(\sqrt{\Delta r}\big)/\Delta xy$,

$$FNE_{\Delta xy,\Delta r}(x,y) = \tanh^2\big((x-y)\cdot m\big)$$

with $\Delta xy > 0,\ 0 < \Delta r < 1.0,\ m > 0$
and $m = \operatorname{artanh}\big(\sqrt{\Delta r}\big)/\Delta xy$,

$$FL_{\Delta xy,\Delta r}(x,y) = \frac{\tanh\big((y-x)\cdot m\big)+1}{2}$$

with $\Delta xy > 0,\ 0 < \Delta r < 0.5,\ m > 0$
and $m = \operatorname{artanh}(2\cdot\Delta r)/\Delta xy$,

$$FG_{\Delta xy,\Delta r}(x,y) = \frac{\tanh\big((x-y)\cdot m\big)+1}{2}$$

with $\Delta xy > 0,\ 0 < \Delta r < 0.5,\ m > 0$
and $m = \operatorname{artanh}(2\cdot\Delta r)/\Delta xy$.

According to Definition 10 and 11 comparisons are written as:

$x \overset{FE}{=} y$	x is fuzzy equal to y
$x \overset{FNE}{\neq} y$	x is fuzzy not equal to y
$x \overset{FL}{<} y$	x is fuzzy less than y
$x \overset{FG}{>} y$	x is fuzzy greater than y

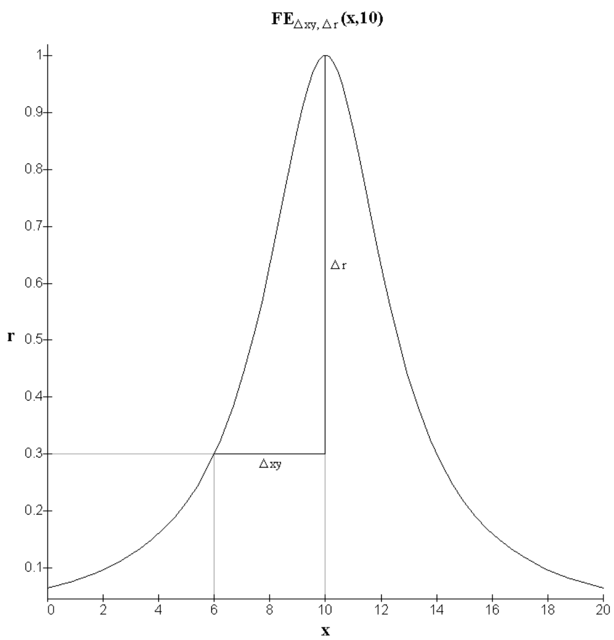

Fig. 1. Result r of the imprecise comparison $x \overset{FE}{=} 10$

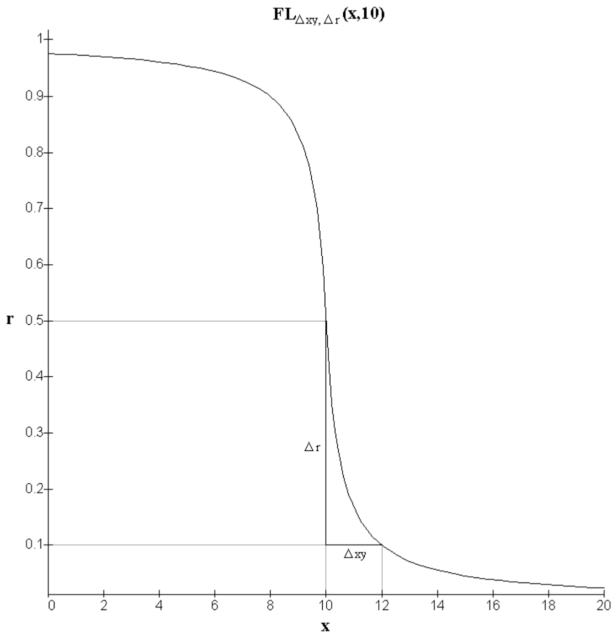

Fig. 2. Result *r* of the imprecise comparison $x \overset{FL}{<} 10$

The parameters Δxy and Δr (respectively *m*, which is calculated by Δxy and Δr) are used to model the "imprecision" of the operators. Fig. 1 shows the setting of Δxy and Δr for *FE* (corresponding setting for *FNE*), Fig. 2 for *FL* (corresponding setting for *FG*).

Determining Δxy and Δr as real values from the source of linguistic verbalized knowledge is quite often difficult, since the knowledge is imprecise. For example, in "about 65 years" the anchor point is given as a real number, but transferring the term "about" into real values Δxy and Δr is sometimes difficult. Following the idea of fuzzy sets we introduce a linguistic slope descriptor to describe the imprecision of the operators. During implementing and maintaining knowledge bases we observed that it is suitable to define between 4 and 6 different "degrees" of imprecision, which led to the following definition.

Definition 12. The parameters Δxy and Δr of the operators *FE* and *FNE* are calculated by the slope descriptor $sd \in \{very\_imprecise, imprecise, ninp, precise, very\_precise\}$ by:

$$\begin{aligned}
sd = very_imprecise &\Rightarrow \Delta xy = 0.49 \cdot (x_{max} - x_{min}) \\
& \Delta r \ = 0.99 \\
sd = imprecise &\Rightarrow \Delta xy = 0.37 \cdot (x_{max} - x_{min}) \\
& \Delta r \ = 0.99 \\
sd = ninp &\Rightarrow \Delta xy = 0.25 \cdot (x_{max} - x_{min}) \\
& \Delta r \ = 0.99 \\
sd = precise &\Rightarrow \Delta xy = 0.13 \cdot (x_{max} - x_{min}) \\
& \Delta r \ = 0.99 \\
sd = very_precise &\Rightarrow \Delta xy = 0.01 \cdot (x_{max} - x_{min}) \\
& \Delta r \ = 0.99
\end{aligned}$$

where *ninp* is the abbreviation for '*neither imprecise nor precise*'.

The parameters Δxy and Δr of the operators *FL* and *FG* are calculated by the slope descriptor $sd \in \{very\_imprecise, imprecise, ninp, precise, very\_precise\}$ by:

$$\begin{aligned}
sd = very_imprecise &\Rightarrow \Delta xy = 0.49 \cdot (x_{max} - x_{min}) \\
& \Delta r \ = 0.49 \\
sd = imprecise &\Rightarrow \Delta xy = 0.37 \cdot (x_{max} - x_{min}) \\
& \Delta r \ = 0.49 \\
sd = ninp &\Rightarrow \Delta xy = 0.25 \cdot (x_{max} - x_{min}) \\
& \Delta r \ = 0.49 \\
sd = precise &\Rightarrow \Delta xy = 0.13 \cdot (x_{max} - x_{min}) \\
& \Delta r \ = 0.49 \\
sd = very_precise &\Rightarrow \Delta xy = 0.01 \cdot (x_{max} - x_{min}) \\
& \Delta r \ = 0.49
\end{aligned}$$

Here, x_{max} is the maximum value of the data under inspection (e.g. 120 years in the example above) and x_{min} is the minimum value (e.g. 0 years). Such a "rough" setting of the imprecision of the comparison operator is good enough for a great majority of problems (see Section 7 for an example). However, sometimes Δxy and Δr have to be *accurately* determined during the process of knowledge acquisition.

The comparison operators have the following important properties:

Theorem 14. *FE* and *FNE* fulfill:

1. The operators are mappings $\Re \times \Re \rightarrow [0,1]$.
2. The operators are commutative, i.e. $FE(x,y) = FE(y,x)$ and $FNE(x,y) = FNE(y,x)$.
3. The operators are reflexive in the sense $FE(x,x) = 1.0$ and $FNE(x,x) = 0.0$.
4. The operators are symmetric in the sense $FE(x-a, x) = FE(x+a, x)$ and $FNE(x-a, x) = FNE(x+a, x)$.
5. The operators are reciprocal symmetric in the sense $FE(x,y) = 1 - FNE(x,y)$.
6. The operators have a "Boolean" behavior for $|x\text{-}y| \rightarrow \infty$:
   $\lim_{|x-y|\to\infty} FE_{\Delta xy,\Delta r}(x,y) = 0$,
   $\lim_{|x-y|\to\infty} FNE_{\Delta xy,\Delta r}(x,y) = 1$.
7. For fixed y and $x \in \Re$:
   $FE(x, y)$ is strictly monotonic ↗ for $x \le y$
   $FE(x, y)$ is strictly monotonic ↘ for $x \ge y$
   $FNE(x, y)$ is strictly monotonic ↘ for $x \le y$
   $FNE(x, y)$ is strictly monotonic ↗ for $x \ge y$
   For fixed x and $y \in \Re$:
   $FE(x, y)$ is strictly monotonic ↗ for $x \le y$
   $FE(x, y)$ is strictly monotonic ↘ for $x \ge y$
   $FNE(x, y)$ is strictly monotonic ↘ for $x \le y$
   $FNE(x, y)$ is strictly monotonic ↗ for $x \ge y$
8. The operators are continuously differentiable functions $\Re^2$.

Proof:

Section 1. – 6. and 8. are already proven in [16].

Interested readers may download this book directly from the internet (link given in the reference section).

7. Fixed y:
   $\forall\, x_1 < x_2 \le y$ satisfy (with $m > 0$):
   $1-\tanh^2((x_1-y)\cdot m) < 1-\tanh^2((x_2-y)\cdot m)$
   $\Rightarrow FE(x, y)$ is strictly monotonically inc.
   $\tanh^2((x_1-y)\cdot m) > \tanh^2((x_2-y)\cdot m)$
   $\Rightarrow FNE(x, y)$ is strictly monotonically dec.
   $\forall\, y \le x_1 < x_2$ satisfy (with $m > 0$):
   $1-\tanh^2((x_1-y)\cdot m) > 1-\tanh^2((x_2-y)\cdot m)$
   $\Rightarrow FE(x, y)$ is strictly monotonically dec.
   $\tanh^2((x_1-y)\cdot m) < \tanh^2((x_2-y)\cdot m)$
   $\Rightarrow FNE(x, y)$ is strictly monotonically inc.
   Fixed x:
   $\forall\, y_1 < y_2 \le x$ satisfy (with $m > 0$):
   $1-\tanh^2((x-y_1)\cdot m) < 1-\tanh^2((x-y_2)\cdot m)$
   $\Rightarrow FE(x, y)$ is strictly monotonically dec.
   $\tanh^2((x-y_1)\cdot m) > \tanh^2((x-y_2)\cdot m)$
   $\Rightarrow FNE(x, y)$ is strictly monotonically inc.
   $\forall\, x \le y_1 < y_2$ satisfy (with $m > 0$):
   $1-\tanh^2((x-y_1)\cdot m) > 1-\tanh^2((x-y_2)\cdot m)$
   $\Rightarrow FE(x, y)$ is strictly monotonically inc.
   $\tanh^2((x-y_1)\cdot m) < \tanh^2((x-y_2)\cdot m)$
   $\Rightarrow FNE(x,y)$ is strictly monotonically dec. □

Theorem 15. *FL* and *FG* fulfill:

1. The operators are mappings $\Re \times \Re \to [0,1]$.
2. The operators are *not* commutative, i.e.
   $FL(x,y) \ne FL(y,x)$ and $FG(x,y) \ne FG(y,x)$
   instead, they are reciprocal commutative, i.e.
   $FL(x,y) = FG(y,x)$.
3. The operators are reflexive in the sense
   $FL(x,x) = 0.5$ and $FG(x,x) = 0.5$.
4. The operators are symmetric in the sense
   $FL(x-a, x) = 1 - FL(x+a, x)$ and
   $FG(x-a, x) = 1 - FG(x+a, x)$.
5. The operators are symmetric in the sense
   $FL(x, y) = 1 - FL(y, x)$ and
   $FG(x, y) = 1 - FG(y, x)$.
6. The operators have a "Boolean" behavior for $|x\text{-}y| \to \infty$:
   $\lim_{(x-y)\to+\infty} FL_{\Delta xy,\Delta r}(x,y) = 0$,
   $\lim_{(x-y)\to-\infty} FL_{\Delta xy,\Delta r}(x,y) = 1$,
   $\lim_{(x-y)\to+\infty} FG_{\Delta xy,\Delta r}(x,y) = 1$,
   $\lim_{(x-y)\to-\infty} FG_{\Delta xy,\Delta r}(x,y) = 0$.
7. For fixed y and $x \in \Re$:
   $FL(x, y)$ is strictly monotonic decreasing
   $FG(x, y)$ is strictly monotonic increasing
   For fixed x and $y \in \Re$:
   $FL(x, y)$ is strictly monotonic increasing
   $FG(x, y)$ is strictly monotonic decreasing
8. The operators are continuously differentiable functions $\Re^2$.

Proof:
Already proven in [16]. □

Property #1 of Theorem 14 and 15, respectively, is important to prove that the operators are imprecise operators in the sense of Definition 10. Property #2, #3 and #6 reflect basic expectations which are derived from the behavior of Boolean comparison operators. The symmetry Properties #4 and #5 probably reflect an intuitive expectation, but we want to point out that in specific areas it might be necessary to have unsymmetrical operators too! Property #7 and #8 are required in order to receive continuous solution spaces without local minima and maxima.

6.3. Combination operators

Humans quite often do not only use isolated conditions for formulating specific problems, but typically combine several conditions. Thus imprecise combination operators (also denominated as extended conjunctors, see Definition 7) are needed to combine single comparisons (set up with the operators *FE*, *FNE*, *FL* and *FG*).

Definition 13. An *imprecise combination operator* $\otimes_{cb}$ combines two or more truth values $w \in [0,1]$:

$\otimes_{cb}(w_1, w_2, \ldots w_n)$: $[0,1]^n \to [0,1]$

which may be also denoted by

$w_1 \otimes_{cb} w_2 \ldots \otimes_{cb} w_n$.

The result of an imprecise combination is again a continuous truth value *r* which lies in the unit interval [0,1].

Many combination operators already exist in Fuzzy Set Theory and other mathematical research work. Since the approach of SFC is slightly different from other Fuzzy methods we identified those of the already existing operators which are suitable to be used in SFC. For aggregation problems (see for example [9], [20], [23]) all existing operators can be used (including t-norms and s-norms). For an averaging combination it turned out, that some operators which have been defined in Fuzzy Set Theory are *not* suitable to work with truth-values $w \in [0,1]$ as in scalar fuzzy control. The Lambda-AND and Lambda-OR [28], which are a special cases of the OWA operator [23], are applicable as shown in [16].

Definition 14. The Lambda operators

AND_λ : $[0,1]^n \to [0,1]$
OR_λ : $[0,1]^n \to [0,1]$

are defined by:

$$\text{AND}_l(w_1, w_2, \ldots, w_n) = l \min_{i=1\ldots n}(w_i) + (1-l)\cdot \overline{w}$$
$$\text{OR}_l(w_1, w_2, \ldots, w_n) = l \max_{i=1\ldots n}(w_i) + (1-l)\cdot \overline{w}$$

with $\lambda \in [0,1]$
and $w_i \in [0,1]$, $i = 1 \ldots n$,

where $\overline{w}$ denotes the mean value of all w_i.

The degree of "imprecision" can be set by choosing appropriate values for the parameter λ. Again, determining λ as a precise value from the source of linguistic verbalized knowledge might be difficult and is quite often not necessary, so we introduce a linguistic slope descriptor for the imprecision of the Lambda operators, too.

Definition 15. The parameter λ of the operators AND_λ and OR_λ is calculated by the slope descriptor *sd* ∈ {*very_imprecise*, *imprecise*, *ninp*, *precise*, *very_precise*} by:

$sd = very\_imprecise \Rightarrow \lambda = 0.00$
$sd = imprecise \Rightarrow \lambda = 0.25$
$sd = ninp \Rightarrow \lambda = 0.50$
$sd = precise \Rightarrow \lambda = 0.75$
$sd = very\_precise \Rightarrow \lambda = 1.00$

Again, *ninp* is the abbreviation for '*neither imprecise nor precise*'.

The Lambda operators have the following important properties:

Theorem 16. AND_λ and OR_λ fulfill:

1. The operators are mappings $[0,1]^n \to [0,1]$.
2. The operators are commutative, i.e.
   $\text{AND}_\lambda(w_1, w_2, \ldots w_n) = \text{AND}_\lambda(w_2, w_1, \ldots w_n) = \ldots$
   $\text{OR}_\lambda(w_1, w_2, \ldots w_n) = \text{OR}_\lambda(w_2, w_1, \ldots w_n) = \ldots$.
3. The operators are idempotent, i.e.
   $\text{AND}_\lambda(w_1, w_1, \ldots w_1) = w_1$ and
   $\text{OR}_\lambda(w_1, w_1, \ldots w_1) = w_1$
4. The operators are continuous functions in $\Re^n$.

Proof:
Already proven in [16]. □

Property #1 proves that the operators are imprecise operators in the sense of Definition 13. Property #2 is again what we typically expect, where Property #3 is not necessarily required but exists. Property #4 is again required in order to receive continuous solution spaces.

Thus, the result *r* of any SFC term is always a truth value, lying in the unit interval [0,1], since the result of imprecise SFC comparisons are truth value, lying in the unit interval, and any Lambda combination of truth values $w \in [0,1]$ lies again in the unit interval.

A typical linguistic verbalization of knowledge is a statement as in

Example 6:

THE MORE data d_1 is equal to antecedent a_1,
AND
THE MORE data d_2 is less than antecedent a_2,
AND
...
THE more the data *r* should be increased .

Each single comparison might be formulated by any term from "*equal*", "*not equal*", "*greater*" or "*less*", and the combination might be formulated by any term from "*and*" or "*or*". Since the knowledge is imprecise, it can be implemented with SFC by

$$r = \left(g_1 \cdot \left(d_1 \otimes_{cp} a_1\right)\right) \otimes_{cb} \left(g_2 \cdot \left(d_2 \otimes_{cp} a_2\right)\right) \ldots \tag{10}$$

where g_i are weighting factors for each single comparison statement (not stated in the linguistic verbalized knowledge in Example 6). From this, one of the major task during knowledge acquisition is to identify the antecedents, the "degrees of imprecision" and the weighting factors. Implementation is then basically a transformation of the linguistic knowledge into SFC formulas.

6.4. SFC being a *Calculus of Imprecise Knowledge*

In this section we prove that scalar fuzzy control is a *Calculus of Imprecise Knowledge* as defined in Section 2. First of all the truth values of SFC follow from Definition 1. Secondly, data used are real constants and real variables, thus they satisfy the corresponding Definitions 2 and 3. We now have to prove that the imprecise comparison operators *FE*, *FNE*, *FL* and *FG* are extended predicates, i.e. they follow the corresponding Definition 6 and satisfy axioms P1 thru P6. Last but not least we have to prove that the imprecise combination operators AND_λ and OR_λ are extended conjunctors, i.e. they follow the corresponding Definition 7 and satisfy axioms J2 thru J4.

Theorem 17. The imprecise comparison operators *FE*, *FNE*, *FL* and *FG* are extended predicates and satisfy the axioms P1 thru P6 of the *Calculus of Imprecise Knowledge*.

Proof:

1. All imprecise comparison operators are mappings
   $\Re \times \Re \to [0,1]$
   as proved in Theorem 14 and 15.
2. *FE* and *FNE* satisfy axiom P1 thru P6 as proved in Theorem 14.
   FL and *FG* satisfy axiom P1 thru P6 as proved in Theorem 15. □

Theorem 18. The imprecise combination operators AND_λ and OR_λ are extended conjunctors and satisfy the axioms J2 thru J4 of the *Calculus of Imprecise Knowledge*.

Proof:

1. AND_λ and OR_λ are mappings
   $[0,1]^n \to [0,1]$
   as proved in Theorem 16.
2. AND_λ and OR_λ satisfy axiom J2 and J4 as proved in Theorem 16.

3. AND_λ and OR_λ satisfy axiom J3:

$$AND_\lambda(w_1,\dots,w_n) =$$
$$\lambda\cdot\min(w_1,\dots,w_n)+(1-\lambda)\cdot\overline{w} =$$
$$\min(w_1,\dots,w_n)+\Delta_{AND\lambda}$$
$$OR_\lambda(w_1,\dots,w_n) =$$
$$\lambda\cdot\max(w_1,\dots,w_n)+(1-\lambda)\cdot\overline{w} =$$
$$\max(w_1,\dots,w_n)-\Delta_{OR\lambda}$$

Let $w_{\min} = \min(w_1, ..., w_n)$
and $w_{\max} = \max(w_1, ..., w_n)$.

3.1 $\Delta_{AND\lambda} = (1-\lambda)\cdot\overline{w}-(1-\lambda)\cdot\min(w_1,\dots,w_n)$

$\Rightarrow\ \Delta_{AND\lambda} = \Delta_{AND\lambda}(w_1,\dots,w_n)$ for given λ .

$\Delta_{OR\lambda} = -(1-\lambda)\cdot\overline{w}+(1-\lambda)\cdot\max(w_1,\dots,w_n)$

$\Rightarrow\ \Delta_{OR\lambda} = \Delta_{OR\lambda}(w_1,\dots,w_n)$ for given λ .

3.2 $\Delta_{AND\lambda}(w,\dots,w) = w-\lambda\cdot w-w+\lambda\cdot w = 0$

$\Delta_{OR\lambda}(w,\dots,w) = -w+\lambda\cdot w+w-\lambda\cdot w = 0$

$\Delta_{AND\lambda} = (1-\lambda)\cdot(\overline{w}-w_{\min}) \ge (1-\lambda)\cdot 0 = 0$

$\Delta_{OR\lambda} = (1-\lambda)\cdot(w_{\max}-\overline{w}) \ge (1-\lambda)\cdot 0 = 0$

$w_{\min} \le \overline{w} \le w_{\max}\ \Rightarrow$

$\Delta_{AND\lambda} \le (1-\lambda)\cdot w_{\max}-(1-\lambda)\cdot w_{\min} < w_{\max}-w_{\min}$

$\Delta_{OR\lambda} \le -(1-\lambda)\cdot w_{\min}+(1-\lambda)\cdot w_{\max} < w_{\max}-w_{\min}$

3.3 Let $w_i \ne \min(w_1, ... w_n)$, $i \in \{1...n\}$
and $w_i < w_i + \delta \le 1 \quad \Rightarrow$

$$\Delta_{AND\lambda}(w_1,\dots,w_i,\dots,w_n) =$$
$$(1-\lambda)\cdot\overline{w}-(1-\lambda)\cdot w_{\min}$$
$$\Delta_{AND\lambda}(w_1,\dots,w_i+\delta,\dots,w_n) =$$
$$(1-\lambda)\cdot\overline{w}'-(1-\lambda)\cdot w_{\min}$$

$\overline{w}$ is the mean value of $w_1,...,w_i,...,w_n$

$\overline{w}'$ is the mean value of $w_1,...,w_i+\delta,...,w_n$

$\Rightarrow\ \overline{w} < \overline{w}'$ and with $0 \le \lambda \le 1 \ \Rightarrow$

$\Delta_{AND\lambda}(w_1, ...w_i, ...w_n) < \Delta_{AND\lambda}(w_1, ...w_i+\delta, ...w_n)$

Let $w_i \ne \max(w_1, ... w_n)$, $i \in 1 ... n$
and $0 \le w_i - \delta < w_i \quad \Rightarrow$

$$\Delta_{OR\lambda}(w_1,\dots,w_i,\dots,w_n) =$$
$$(\lambda-1)\cdot\overline{w}+(1-\lambda)\cdot w_{\max}$$
$$\Delta_{OR\lambda}(w_1,\dots,w_i+\delta,\dots,w_n) =$$
$$(\lambda-1)\cdot\overline{w}'+(1-\lambda)\cdot w_{\max}$$

$\overline{w}$ is the mean value of $w_1,...,w_i,...,w_n$

$\overline{w}'$ is the mean value of $w_1,...,w_i-\delta,...,w_n$

$\Rightarrow\ \overline{w} > \overline{w}'$ and with $0 \le \lambda \le 1 \ \Rightarrow$

$\Delta_{OR\lambda}(w_1, ...w_i, ...w_n) < \Delta_{OR\lambda}(w_1, ...w_i-\delta, ...w_n)$ □

Theorem 18 shows that the imprecise combination operators AND_λ and OR_λ are *classes* of extended conjunctors. The parameter λ defines the "degree of averaging", i.e. for each λ the combination operators build up a specific extended conjunctor.

Finally, the "De' Morgan law for extended conjunctors" is valid for AND_λ and OR_λ, since AND_λ and OR_λ satisfy the requirement of the corresponding theorem:

Theorem 19. AND_λ and OR_λ satisfy

$\Delta_{AND\lambda}(w_1, ... , w_n) = \Delta_{OR\lambda}(1-w_1, ... , 1-w_n)$

$\Delta_{OR\lambda}(w_1, ... , w_n) = \Delta_{AND\lambda}(1-w_1, ... , 1-w_n)$

for $\forall\ w_1, ... , w_n \in [0,1]$.

Proof:

Let $w_{\min} = \min(w_1,... w_n)$
and $w_{\max} = \max(w_1,... w_n)$

$$\overline{w} = \frac{\sum_{i=1}^{n} w_i}{n} \qquad \text{and}$$

$$\overline{1-w} = \frac{\sum_{i=1}^{n} 1-w_i}{n} = \frac{\sum_{i=1}^{n}1-\sum_{i=1}^{n} w_i}{n} = \frac{n}{n}-\frac{\sum_{i=1}^{n} w_i}{n} = 1-\overline{w}$$

$\Delta_{AND\lambda}(w_1, ... w_n) = (1-\lambda)\cdot\overline{w} - (1-\lambda)\cdot\min(w_1, ... w_n)$

$\Delta_{OR\lambda}(w_1, ... w_n) = -(1-\lambda)\cdot\overline{w} + (1-\lambda)\cdot\max(w_1, ... w_n)$

1. $\Delta_{OR\lambda}(1-w_1,\dots 1-w_n) =$

$$-(1-\lambda)\cdot\overline{1-w}+(1-\lambda)\cdot\max(1-w_1,\dots 1-w_n) =$$
$$-(1-\lambda)\cdot(1-\overline{w})+(1-\lambda)\cdot\max(1-w_1,\dots 1-w_n) =$$
$$-(1-\lambda)\cdot(1-\overline{w})+(1-\lambda)\cdot(1-w_{\min}) =$$
$$(1-\lambda)\cdot\overline{w}-(1-\lambda)\cdot w_{\min} =$$
$$\Delta_{AND\lambda}(w_1,\dots w_n)$$

2. $\Delta_{AND\lambda}(1-w_1,\dots 1-w_n) =$

$$(1-\lambda)\cdot\overline{1-w}-(1-\lambda)\cdot\min(1-w_1,\dots 1-w_n) =$$
$$(1-\lambda)\cdot(1-\overline{w})-(1-\lambda)\cdot\min(1-w_1,\dots 1-w_n) =$$
$$(1-\lambda)\cdot(1-\overline{w})-(1-\lambda)\cdot(1-w_{\max}) =$$
$$(1-\lambda)\cdot\overline{w}+(1-\lambda)\cdot w_{\max} =$$
$$\Delta_{OR\lambda}(w_1,\dots w_n)$$

□

7. A SFC solution for the inverted pendulum problem

Scalar fuzzy control has already been used and investigated in a few research and engineering projects. [15] shows first results of our research work on expert systems for programming implantable devices such as cardiac pacemakers and implantable defibrillators. In [17] we discuss possibilities to use SFC methods in the area of aggregation problems, where the task is to gather confidence in given hypotheses. Furthermore SFC was used to successfully stabilize and drive a highly dynamic fuel cell [18].

In this section we demonstrate the usage of the *Calculus of Imprecise Knowledge* and SFC on the simple and well known problem of the inverted pendulum (see for example [3], [4], [14], [19], [22]). By contrast to above mentioned applications this problem is simple enough to be completely described in this paper. A comparison of the SFC solution presented here with other suggested models for the inverted pendulum would exceed the scope of this paper.

7.1. The inverted pendulum system

The task of the inverted pendulum problem is both to stabilize the pendulum in an upright position *and* keep the cart at the origin position or drive the cart always back to the origin position, respectively.

The inverted pendulum system was simulated on a computer by using Eqs. (11) and (12).

$$\begin{aligned}(m_c + m_p)x''(t) = \; & -c_m x'(t) \\ & + F(t) \\ & - m_p l e''(t)\cos(e(t)) \\ & + m_p l e'(t)^2 \sin(e(t))\end{aligned} \tag{11}$$

$$4/3\ l\ e''(t) + x''(t)\cos(e(t) - g\sin(e(t)) = 0 \tag{12}$$

Here, e is the pendulum angle, where e=0 corresponds to an upright position, e>0 corresponds to pendulum deflections to the right and e<0 corresponds to pendulum deflections to the left. x is the cart position, with x=0 if the cart is located at the origin position, x>0 if the cart is right from the origin and x<0 if the cart is left from the origin position. F is the force driving the cart-pendulum system, thus stabilizing the pendulum in an upright position and keeping the cart at the origin position. It is the task of the implementation to calculate F to obtain a stabilized system.

Table 1 lists all variables and gives the parameter setting used for the simulation.

7.2. Knowledge base

For a human being it is quite simple to solve the inverted pendulum problem, as an artist can easily demonstrate. Nevertheless, asked for a mathematical description of the solution path, the great majority of persons will reply that they can not provide it. Instead, it is easy to acquire the linguistic description of the knowledge used to solve the problem. A possible acquisition result is shown in

Example 7:

Knowledge on the inverted pendulum problem

Table 1
Inverted Pendulum Parameters

Parameter	Symbol	Value
Pendulum angle	e	in rad
Pendulum angular velocity	$\dot{e}$	in rad/s
Cart position	x	in m
Cart velocity	$\dot{x}$	in m/s
Force to drive the cart	F	in N
Mass of cart	m_c	1.0 kg
Friction of cart	c_μ	0.2 m
Mass of pole	m_p	0.1 kg
Length of pole	l_p	1.0 m
Step size for the simulation	Δ_T	0.02 s

1) Stabilizing the pendulum:
   - Rule #1: The more the pole is deflected to the right (positive pole angle) the greater should be the driving force F (positive F) to prevent the pendulum from falling completely to the right side.
     Similarly for pole deflections to the left.
   - Rule #2: Additionally to rule #1, the more the angular velocity is positive *and* the pole angle is *not* deflected much to the left, the more should F be increased.
     This will prevent a pole, which is *not* deflected to the left and simultaneously rapidly deflecting to the right, from falling completely to the right side.
     Similarly for negative angular velocities.
2) Keeping the cart at or driving the cart back to the origin position:
   - Rule #3: Additionally, the more the cart is positioned to the right (far away from the origin), the more F should be *slightly* increased.
     This will cause the pole to "fall" slightly to the left, thus allowing the cart to accelerate slightly to the left and thus maneuver it back to the origin position.
     Similarly for negative angular velocities.
   - Rule #4: Additionally, the more the pendulum is stabilized *and* the more the cart velocity is negative (cart moves to the left) *and* the more the cart is near to the origin (or already left of the origin), the more F should be decreased. This will cause the pole to "fall back" to the right, thus slowing down the whole system.
     Similarly for positive cart velocities.

Before transforming the linguistic verbalized knowledge into an SFC implementation, the expert has to refine the knowledge to receive the specific antecedents as well as the required degree of imprecision. For example for rule #1 the question to the expert is, which pole angle is supposed to be great already, thus needing quick countermeasures. Certainly, a pole angle of e.g. 0.05 rad does not need an immediate countermeasure. Experience shows that from about 0.35 rad (approximately 20 degree) countermeasures could be started. Countermeasures should be started *moderately*, thus the imprecision of the comparison is '*ninp*'.

Similarly, the refinement can be achieved for rules #2 – 4. We thus receive the following SFC implementation, as shown in formula (13). The lines of (13) correspond to the eight rules of example 7 (first line implements the first sentence of rule #1, second line implements the second sentence of rule #1, etc.).

$$\begin{aligned}F \;=\; & +100\cdot\left(e \overset{FG}{\underset{ninp}{>}} +0.35\right) \\ & -100\cdot\left(e \overset{FL}{\underset{ninp}{<}} -0.35\right)\end{aligned}$$

$$
\begin{aligned}
&+100\cdot\left(\left(\dot{e}\overset{FG}{\underset{ninp}{>}}+1.00\right)\ \overset{F}{\underset{precise}{AND}}\ \left(e\overset{FG}{\underset{ninp}{>}}-0.5\right)\right)\\
&-100\cdot\left(\left(\dot{e}\overset{FL}{\underset{ninp}{<}}-1.00\right)\ \overset{F}{\underset{precise}{AND}}\ \left(e\overset{FL}{\underset{ninp}{<}}+0.5\right)\right)\\
&+\ \ 25\cdot\left(x\overset{FG}{\underset{ninp}{>}}+3\right)\qquad\qquad(13)\\
&-\ \ 25\cdot\left(x\overset{FL}{\underset{ninp}{>}}-3\right)\\
&-100\cdot\left(\left(|\dot{e}|\overset{FL}{\underset{ninp}{<}}0.1\right)\overset{F}{\underset{ninp}{AND}}\left(\dot{x}\overset{FL}{\underset{ninp}{<}}-5\right)\overset{F}{\underset{ninp}{AND}}\left(x\overset{FL}{\underset{ninp}{<}}+3\right)\right)\\
&+100\cdot\left(\left(|\dot{e}|\overset{FL}{\underset{ninp}{<}}0.1\right)\overset{F}{\underset{ninp}{AND}}\left(\dot{x}\overset{FL}{\underset{ninp}{<}}-5\right)\overset{F}{\underset{ninp}{AND}}\left(x\overset{FL}{\underset{ninp}{<}}+3\right)\right)
\end{aligned}
$$

As mentioned above we obtain one SFC rule per linguistic rule keeping the rulebase small and simple.

Driving the pendulum cart system this rulebase is to be continuously used to determine the force F momentarily needed to stabilize the system, i.e. F is continuously calculated by the rulebase (13). The inputs are real values (e, $\dot{e}$, x, $\dot{x}$) and the output (F) is directly calculated, i.e. $F = F(e,\dot{e},x,\dot{x})$ in accordance to Eq. (13). In other words: fuzzification, aggregation and defuzzification in fuzzy control are replaced by SFC rules in scalar fuzzy control.

For the simulation the system response for a given force F was calculated by Eqs. (11) and (12). With a step size of Δ_T = 0.02s, the force F as well as the system response was recalculated every 20 ms.

Figs. 3, 4 and 5 show three step responses with different initial conditions. The proposed SFC implementation obviously leads to a system which is able to quickly stabilize the inverted pendulum through a wide range of different initial conditions, even for very big pendulum deflections as in Figs. 3 and 5.

To demonstrate the robustness of the solution, the antecedents have been varied by ±10% from the values given above and the slope descriptors have been varied in between *imprecise* and *ninp*. Fig. 6 shows the result for all possible combinations of varied antecedents and slope descriptors. All possible knowledge implementations, set up by all combinatorial arrangements, result in a clear stabilization of the pendulum-cart system. Clearly, selecting antecedents and/or slope descriptors far away from the refined values will result in an oscillating or unstable system, since it does no longer represent the expert knowledge.

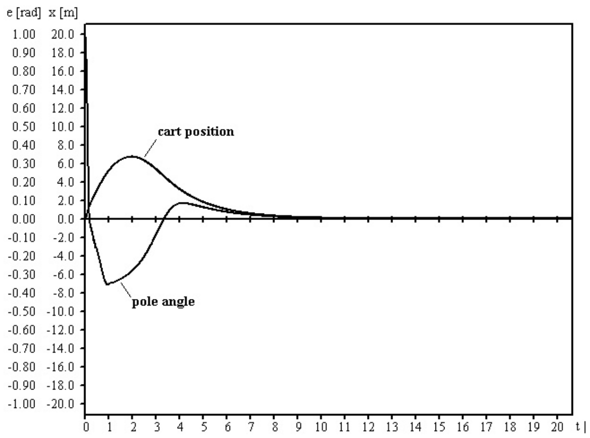

Fig. 3. Response for e = 1 rad and x = 0 m

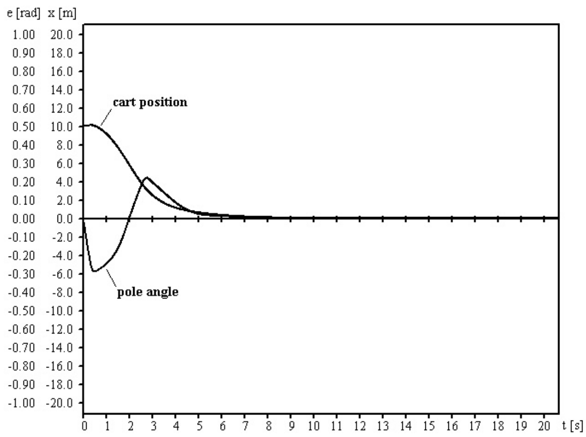

Fig. 4. Response for e = 0 rad and x = 10 m

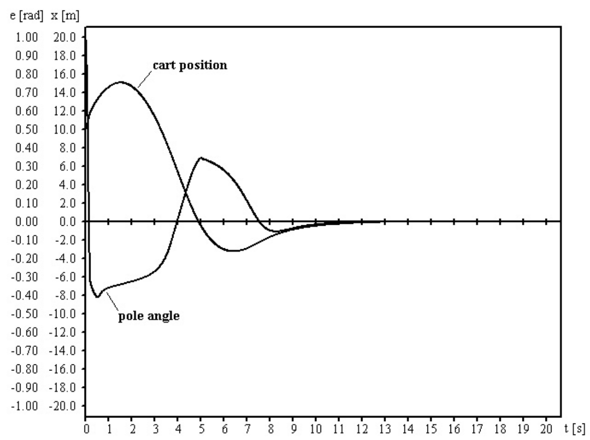

Fig. 5. Response for e = 1 rad and x = 10 m

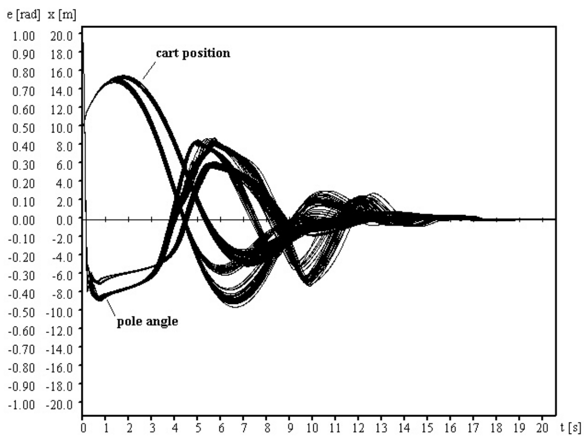

Fig. 6. Response for e = 1 rad and x = 10 m for different antecedents and slope descriptors

8. Conclusion

While it is difficult for humans to “translate” linguistic statements into expressions on an absolute scale level, it is quite easy for humans to “translate” them into comparative statements. The *Calculus of Imprecise Knowledge* introduced in this paper provides the possibility to represent such comparative statements by means of the extended predicates and conjunctors. From this, the *Calculus of Imprecise Knowledge* may be understood as a kind of comparative logic, too. It is suitable to easily represent and implement imprecise statements and knowledge.

While the *Calculus of Imprecise Knowledge* gives an axiomatic framework, scalar fuzzy control (SFC) is one suggestion for an application oriented specialization. It is suitable in all areas, where imprecise knowledge deals with real valued data. The basics of SFC is that no linguistic variables are to be defined. From this, data are *not* to be fuzzified, aggregated and defuzzified again. Furthermore it is thus not necessary to choose appropriate methods for the fuzzification, aggregation and defuzzification processes. In SFC the knowledge is implemented in terms of SFC rules instead, which use imprecise comparison and combination operators. These SFC rules are used to directly calculate the output data on the basis of given input data.

Scalar fuzzy control has been used in a few research and engineering projects. [15] shows first results of our research work on expert systems for programming implantable devices such as cardiac pacemakers and implantable defibrillators. In [17] we discuss possibilities to use SFC methods in the area of aggregation problems, where the task is to gather confidence in given hypotheses. Furthermore SFC was used to successfully stabilize and drive a highly dynamic fuel cell [18].

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